Erratum to the paper
"Non-trapping condition for semiclassical Schrödinger operators with matrix-valued potentials."
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I Corrections.

In the paper mentioned above, the proof of Theorem 1.4 is incorrect. Using the notation of the paper, we shall precisely explain what is incorrect. We do not know if Theorem 1.4 holds true but we shall give a weaker version of it, the proof of which is availlable in the present Section II. Some other mistakes, detected by the referee, are also corrected.

Let us begin with the mistakes, which take place near (4.7) on page 26. The text begining with the sentence containing (4.7) and ending on the same page with a formula containing integrals (just before the words "Notice that") should be replaced by the following one:

"It is the unitary operator on $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$ given by

$$c(\mathrm{x}_0^*) \;=\; U_h\, \exp\Big(ih^{-1/2}(x\cdot\xi_0 - x_0\cdot D_x)\Big)\,\mathrm{I}_m\,, \tag{4.7}$$

where U_h acts on $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$ by $U_hf(x) = h^{-d/4}f(h^{-1/2}x)$ (cf. [R, J2]). For any $S\in\Sigma_{r,t}$ with $r,t\leq 0$ (cf. (3.1)), $c(\mathrm{x}_0^*)^*S_h^w c(\mathrm{x}_0^*) = S(\mathrm{x}_0^*) + O_S(h^{1/2})$, strongly in $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$. Since $\theta(\lambda)=1$, we obtain, for $\mathrm{x}_0^*\in E(\lambda)$,

$$\int_{-T}^{T} c(\mathrm{x}_0^*)^*\theta\Big(\hat{P}(h)\Big)\Big(F^t(B)\Big)_h^w\theta\Big(\hat{P}(h)\Big)c(\mathrm{x}_0^*)\,dt \;=\; \int_{-T}^{T} F^t(B)(\mathrm{x}_0^*)\,dt \,+\, O_T(h^{1/2})\,,$$

strongly in $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$.".

The mentioned error is located in Section 4. The formulation of Kato's notion of locally $\hat{P}(h)$-smoothness in (4.1) is false. We are very grateful to Monique Combescure who noticed this error. As a consequence, the proof Theorem 1.4 in Section 4.2, which crucially uses (4.1), does not work.

Since, for all t,

$$\begin{aligned}\Big\|\langle x\rangle^{-s}\,\theta\Big(\hat{P}(h)\Big)\,U_h(t)\Big\|^2 &= \Big\|\Big(\langle x\rangle^{-s}\,\theta\Big(\hat{P}(h)\Big)\,U_h(t)\Big)\cdot\Big(\langle x\rangle^{-s}\,\theta\Big(\hat{P}(h)\Big)\,U_h(t)\Big)^*\Big\| \\ &= \Big\|\langle x\rangle^{-s}\,\theta\Big(\hat{P}(h)\Big)^2\,\langle x\rangle^{-s}\Big\|\,,\end{aligned}$$

the integral in (4.1) diverges! The correct formulation of Kato's notion of locally $\hat{P}(h)$-smoothness is the following. There exists $C_s>0$ such that, for all $f\in\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$,

$$\int_{\mathbb{R}}\Big\|\langle x\rangle^{-s}\,\theta\Big(\hat{P}(h)\Big)\,U_h(t)f\Big\|^2\,dt\;\leq\;C_s\|f\|^2\,. \tag{I.1}$$

Replacing (4.1) by this estimate (I.1), the rest of Section 4.1 is valid (with the corrections given above). In particular, the proofs of Proposition 4.1 and Theorem 1.8 are correct. We do not see how to repare the proof of Theorem 1.4 in Section 4.2. However, we shall prove the following weaker version of Theorem 1.4.

Theorem I.1. *Consider a model of codimension* 1 *crossing that satisfies the special condition at the crossing and the structure condition at infinity. Let $\lambda>\|M_\infty\|_m$. If the property* (1.3) *and* (1.4)) *holds true then λ is non-trapping for all eigenvalues of the symbol P of $\hat{P}(h)$.*

Remark I.2. *If $d=1$ (i.e. if the variable x lies in $\mathbb{R}$), we do not need the special condition at the crossing to prove Theorem I.1 of the present erratum, as shown in Remark II.1.*

Remark I.3. *The comments at the end of Section 5 on the role played by the special condition at the crossing are relevant for the proof of the present Theorem I.1.*

II Proof of Theorem I.1.

Let $T>0$. Under the hypotheses of Proposition 4.3 but for an arbitrary symbol $B\in\Sigma_{0,0}$, we follow the beginning of its proof until (4.11) with $t\in[-T;T]$. Putting all together, we obtain, for such t,

$$\begin{aligned}&\chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\Big(U_h(t)^*B_h^wU_h(t)\;-\;\Big(F^t(B)\Big)_h^w\Big)\chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\\ =\;&\sum_{q=0}^{p}R_q\;+\;\tilde{O}_T(\epsilon)\;+\;\tilde{O}_{T,\epsilon}(h)\,,\end{aligned} \tag{II.1}$$

where

$$R_q \;=\; -\int_0^t \chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\, U_h(t-r)^*\, \Big(F^r(B)\Big)^w_h \cdot \Bigg(\sum_{\substack{j,k,l\in I\\ j\neq k}} \Pi_j\Big(2\xi\cdot\nabla\Pi_l\Big)\Pi_k\,\psi_q\Bigg)^w_h U_h(t-r)\,\chi\Big(\hat{P}(h);\lambda,\epsilon\Big)\,dr\,. \tag{II.2}$$

Now we introduce coherent states. Let $\mathrm{x}_0^*\in E^*(\lambda)$ and $f\in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$ with $\|f\|=1$. Then $f_h := c(\mathrm{x}_0^*)f$ (cf. (4.7)) is a coherent state microlocalized near x_0^*. Since $\chi(\hat{P}(h);\lambda,\epsilon)f_h = f_h + O_\epsilon(h^{1/2})$ in $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$,

$$\Big\langle f_h\,,\,U_h(t)^*B_h^wU_h(t)f_h\Big\rangle \;-\; \Big\langle f_h\,,\,\Big(F^t(B)\Big)^w_h f_h\Big\rangle \;=\; \sum_{q=0}^p r_q \;+O_T(\epsilon)+O_{T,\epsilon}(h^{1/2})\,, \tag{II.3}$$

for $t\in[-T;T]$, where

$$r_q \;=\; -\int_0^t \Big\langle f_h\,,\,U_h(t-r)^*\,\Big(F^r(B)\Big)^w_h \cdot \Bigg(\sum_{\substack{j,k,l\in I\\ j\neq k}} \Pi_j\Big(2\xi\cdot\nabla\Pi_l\Big)\Pi_k\,\psi_q\Bigg)^w_h U_h(t-r)f_h\Big\rangle\,dr\,. \tag{II.4}$$

We choose a polarized coherent state f_h by requiring that, for some $k_0\in\{1;\cdots;N\}$, $\Pi_{k_0}(x_0)f=f$. In particular, $\Pi_jf_h=\delta_{jk_0}f_h+O(h^{1/2})$ in $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$. Let $\eta>0$. If the trajectory $(\phi^t_{k_0}(\mathrm{x}_0^*))_{t\in[-T;T]}$ does not intersect the set

$$\mathcal{C}^*(\lambda;\eta) \;:=\; \Big\{\mathrm{x}^*=(x,\xi)\in\mathcal{C}^*(\lambda)\,;\,|\xi\cdot\nabla\tau(x)|\;<\;\eta\Big\}\,, \tag{II.5}$$

then, by [H],

$$\Pi_jU_h(t)f_h \;=\; \delta_{jk_0}U_h(t)f_h \;+\; O_{T,\eta}(h^{1/2}) \tag{II.6}$$

in $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^m)$, for $t\in[-T;T]$. In particular, for any q, $r_q=O_{T,\eta}(h^{1/2})$, by (II.4), yielding

$$\Big\langle f_h\,,\,U_h(t)^*B_h^wU_h(t)f_h\Big\rangle \;-\; \Big\langle f_h\,,\,\Big(F^t(B)\Big)^w_h f_h\Big\rangle \;=\; O_T(\epsilon)\;+\;O_{T,\epsilon,\eta}(h^{1/2})\,, \tag{II.7}$$

for $t\in[-T;T]$. Now, we assume that the open set

$$J(\mathrm{x}_0^*;\eta) \;=\; \Big\{t\in[-T;T];\,\phi^t_{k_0}(\mathrm{x}_0^*)\in\mathcal{C}^*(\lambda;\eta)\Big\}$$

is not empty and we want to show

$$\Big\langle f_h\,,\,U_h(t)^*B_h^wU_h(t)f_h\Big\rangle \;-\; \Big\langle f_h\,,\,\Big(F^t(B)\Big)^w_h f_h\Big\rangle \;=\; O_T(\eta)\;+\;O_T(\epsilon)\;+\;O_{T,\epsilon,\eta}(h^{1/2})\,, \tag{II.8}$$

for $t\in[-T;T]$. If $0\not\in J(\mathrm{x}_0^*;2\eta)$ and $t_0,t_1>0$ such that $\phi^t_{k_0}(\mathrm{x}_0^*)\not\in\mathcal{C}^*(\lambda;\eta)$ for $t\in[t_0;t_1]$, then, setting $g_h:=c(\phi^{t_0}_{k_0}(\mathrm{x}_0^*))f$,

$$\Big\langle f_h\,,\,U_h(t)^*B_h^wU_h(t)f_h\Big\rangle \;-\; \Big\langle g_h\,,\,U_h(t-t_0)^*B_h^wU_h(t-t_0)g_h\Big\rangle \;=\; O_T(\epsilon)\;+\;O_{T,\epsilon,\eta}(h^{1/2})\,,$$

for $t \in [t_0; t_1]$, by (II.7) applied to the f_h and to the g_h. Because of this fact and of (II.7), it is sufficient to get (II.8) on $[-T; T]$ to show it on $[0; t_0[$ in the following case: $[0; t_0[\subset J(\mathrm{x}_0^*; 2\eta)$ and $t_0 \not\in J(\mathrm{x}_0^*; 2\eta)$. Notice that t_0 may be bigger than T and even infinite.
Let $\tilde{\psi}, \tilde{\psi}_1 \in C_0^\infty(T^*\mathbb{R}^d; \mathbb{R})$ such that $\tilde{\psi} = 1$ on $\mathcal{C}^*(\lambda; 3\eta)$, $\operatorname{supp} \tilde{\psi} \subset \mathcal{C}^*(\lambda; 4\eta)$, and $\tilde{\psi}^2 + \tilde{\psi}_1^2 = 1$. Take $B = \tilde{\psi}_1^2 \mathrm{I}_m$ and $t \in [0; \min(t_0, T)[$. Let $A_j(t) := (\tilde{\psi}_1)_h^w \Pi_j U_h(t) f_h$. For any q, there exists a continuous function $r \mapsto \alpha_q(r) \in \mathbb{R}^+$ such that

$$|r_q| \ \leq \ \int_0^t \Big(\sum_j \|A_j(t-r)\|^2\Big)^{1/2} \alpha_q(r) \Big(\sum_k \|A_k(t-r)\|^2\Big)^{1/2} dr \ + \ O_T(h) \, ,$$

since the symbols of the Weyl h-pseudodifferential operators in (II.4) are bounded. Putting all together and setting $z(t;h) := \langle f_h \, , \, U_h(t)^* B_h^w U_h(t) f_h \rangle$, we obtain

$$z(t;h) \ \leq \ \Big\langle f_h \, , \, \Big(F^t(B)\Big)_h^w f_h \Big\rangle \ + \ \int_0^t z(t-r;h) \sum_q \alpha_q(r) \, dr \ + \ O_T(\epsilon) \ + \ O_{T,\epsilon,\eta}(h^{1/2}) \, .$$

By Gronwall's lemma (see [DG] for instance),

$$z(t;h) \ \leq \ e^{O_T(h^0)} \Big(\Big\langle f_h \, , \, \Big(F^t(B)\Big)_h^w f_h \Big\rangle \ + \ O_T(\epsilon) \ + \ O_{T,\epsilon,\eta}(h^{1/2}) \Big) \ .$$

Since f_h is polarized,

$$\begin{aligned} \Big\langle f_h \, , \, \Big(F^t(B)\Big)_h^w f_h \Big\rangle \ &= \ \Big\langle f_h \, , \, \Big(F^t((1-\tilde{\psi})^2 \Pi_{k_0})\Big)_h^w f_h \Big\rangle \ + \ O_T(h) \\ &= \ \tilde{\psi}_1^2 \circ \phi_{k_0}^t(\mathrm{x}_0^*) \ + \ O_T(h^{1/2}) \ = \ O_T(h^{1/2}) \, , \end{aligned}$$

for $t \in [0; \min(t_0, T)[$, since $[0; \min(t_0, T)[\subset J(\mathrm{x}_0^*; 2\eta)$. This yields $\|(\tilde{\psi}_1)_h^w U_h(t) f_h\|^2 = O_T(\epsilon) \ + \ O_{T,\epsilon,\eta}(h^{1/2})$. For arbitrary B, $t \in [0; \min(t_0, T)[$, and all q,

$$\begin{aligned} r_q \ &= \ - \int_0^t \Big\langle f_h \, , \, U_h(t-r)^* \Big(F^r(B)\Big)_h^w \\ &\quad \cdot \Big(\sum_{\substack{j,k,l \in I \\ j \neq k}} \Pi_j \Big(2\xi \cdot \nabla \Pi_l\Big) \Pi_k \, \psi_q \tilde{\psi}^2 \Big)_h^w U_h(t-r) f_h \Big\rangle \, dr \ + \ O_T(\epsilon) \ + \ O_{T,\epsilon,\eta}(h^{1/2}) \\ &= \ O_T(\eta) \ + \ O_T(\epsilon) \ + \ O_{T,\epsilon,\eta}(h^{1/2}) \, , \end{aligned} \tag{II.9}$$

since $\xi \cdot \nabla \Pi_l = O(\eta)$ on the support of $\tilde{\psi}$ by the special condition at the crossing (cf. Definition 2.9). Thus (II.3) yields (II.8).
Now the estimate (I.1) with f replaced by f_h, (II.7), and (II.8) imply that, for $B = \langle x \rangle^{-2x} \mathrm{I}_m$,

$$\int_{-T}^T \Big\langle f_h \, , \, \Big(F^t(B)\Big)_h^w f_h \Big\rangle \, dt \ \leq \ C_s + O_{T,\eta,\epsilon}(h^{1/2}) \ + \ O_T(\eta) \ + \ O_T(\epsilon) \ \leq \ 2C_s \ + \ O_{T,\eta,\epsilon}(h^{1/2}) \, ,$$

for some η, ϵ small enough. Since $\langle f_h \, , \, (F^t(B))_h^w f_h \rangle \ = \ \langle \pi_x \phi_{k_0}^t(\mathrm{x}_0^*) \rangle^{-2s} + O_T(h^{1/2})$, we obtain, letting $h \to 0$,

$$\int_{-T}^T \langle \pi_x \phi_{k_0}^t(\mathrm{x}_0^*) \rangle^{-2s} dt \ \leq \ 3C_s \, .$$

As in Section 4.2, this yields the non-trapping condition for p_{k_0} at energy λ.

Remark II.1. *If $d = 1$, the above proof gives a better result. Indeed, the function $\tilde{\psi}$ actually localizes near $\xi = 0$ so we directly get (II.9). We did not use the special condition at the crossing.*