ORDINARY DIFFERENTIAL EQUATIONS – F01, L3

Exercises – Week 1.

Exercice I (Vector product)

The space $\mathbb{R}^3$ is endowed with the canonical scalar product $(\cdot,\cdot)$.

1) Give the definition of the vector product $a \times b$ of two vectors $a, b \in \mathbb{R}^3$ and its geometrical interpretation.

2) Show the formula

$$\forall x, y, z \in \mathbb{R}^3, \quad x \times (y \times z) = (x, z)y - (x, y)z.$$

3) Deduce the Jacobi rule

$$\forall x, y, z \in \mathbb{R}^3, \quad x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0.$$

4) Let $a, b \in \mathbb{R}^3$, $a \neq 0$. Give a necessary and sufficient condition for the solvability of the equation

$$a \times x = b. \tag{1}$$

5) Under this condition, show that the set of solutions of (1) is

$$\left\{ \frac{1}{\|a\|^2} b \times a + \lambda a; \lambda \in \mathbb{R} \right\}.$$

Exercise II (Conics in polar coordinates)

In the affine plane P is given a straight line D, a point $F \notin D$, and a positive real number e. By definition, the conic $\mathcal{C}$ of directrix D, focus F and excentricity e is the set of point M of P such that

$$d(M, F) = ed(M, P).$$

We let $(O, \mathbf{i}, \mathbf{j})$ be an orthonormal affine base in P in which the equation of D is $x = d$ and in which $F = (c, 0)$ with $c > 0$. The origin O can also be chosen in such a way that $d/c = e^2$.

1) Show that, in $(O, \mathbf{i}, \mathbf{j})$, the equation of $\mathcal{C}$ is

$$(x - c)^2 + y^2 = e^2 (x - d)^2 .$$

Set $a := c/e$; show that the equation rewrites

$$x^2 \left(\frac{a^2 - c^2}{a^2} \right) + y^2 = a^2 - c^2. \tag{2}$$

2) Show that, in $(F, \mathbf{i}, \mathbf{j})$, the equation of $\mathcal{C}$ in polar coordinates (r, θ) is

$$r(1 + \varepsilon e \cos(\theta)) = \varepsilon p \tag{3}$$

where $p = e(d - c)$ and $\varepsilon \in \{-1, +1\}$.

Assume $p > 0$. Using the fact that the point $(r = p, \theta = \pi/2)$ belongs to $\mathcal{C}$ (why ?), precise the accurate sign in (3).

3) Using (2)-(3), show that

3-a) If $e = 0$, the conic is a circle of center O and radius a.

3-b) If $0 < e < 1$, the conic is an ellipse.

3-c) For $e = 1$, the conic is a parabola.

3-d) For $e > 1$, the conic is an hyperbola.

Exercise III (Speed of escape of the attraction of earth)

A satellite S is launched from the surface of the earth. Let r denote its distance from the center of the earth and v the magnitude of its speed.

1) Show that

$$\frac{1}{2}v^2 - \frac{\kappa}{r}$$

is a constant of the evolution (κ denotes the universal constant of the gravitation $\kappa = 6,67.10^{-11} S.I.$).

2) Deduce at which minimal initial speed the satellite will escape the attraction of the earth and get lost in the space (radius of the earth: $6380km$).

Exercice IV (Antiderivatives, ODE with separate variables)

Solve the following ordinary differential equations:

$$\dot{y} = y^\alpha \quad (\alpha \in \mathbb{R}), \qquad \dot{y} = (1-y)y, \qquad \dot{y} = \tan(t)y,$$

$$\dot{y} = \frac{\pi}{4}\cos(t)(1+y^2), \qquad \dot{y} = t\sqrt{1-y^2}, \qquad \dot{y} = 2^t \cos^2(y).$$

Exercice V (Scalar first order linear ODE)

Solve the following ordinary differential equations:

$$\dot{y} + y = \cos(t), \qquad (1+t^2)\dot{y} = 2ty + 5(1+t^2), \qquad (1+t^2)\dot{y} = ty + 5(1+t^2),$$

$$\dot{y} - y = e^{\alpha t} \quad (\alpha \in \mathbb{R}).$$

Exercice VI (Dynamics of populations)

We are interested in the evolution of a population. Let $x(t)$ be the number of living people in the population at the times t and let k be the *growth rate* of the population defined by $k = \dfrac{\dot{x}(t)}{x(t)}$.

Analyse the evolution of the population in each following case:

1) (Normal reproduction) Case $k = constant$. Discuss according to the sign of k.

2) (Explosive equation) Case $k = x$. Show the blow-up in finite time.

3) (Logistic model) Case $k = a - bx$. Show that the population goes to an equilibrium.

4) Draw the trajectories of the ordinary differential equations $\dot{x} = kx$ in each preceding case on the following pictures.

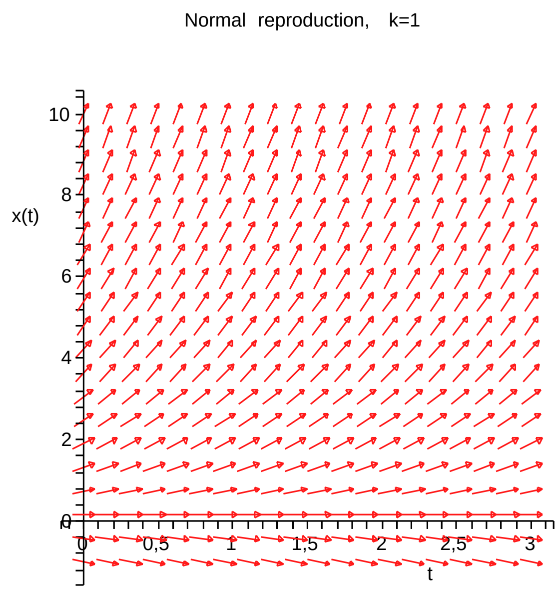

Figure 1: Normal Reproduction

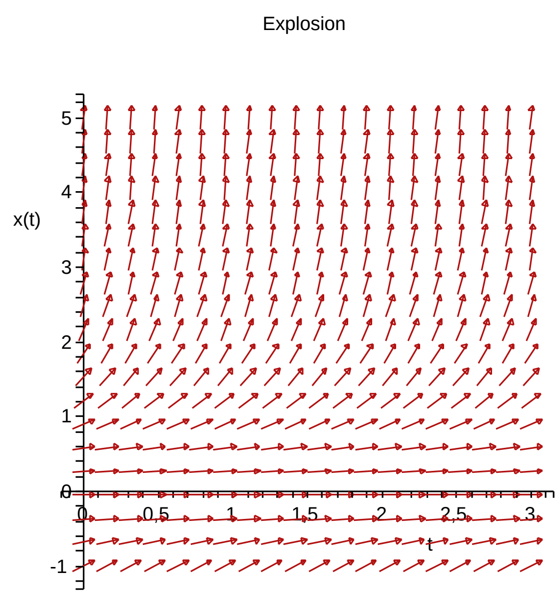

Figure 2: Explosive equation

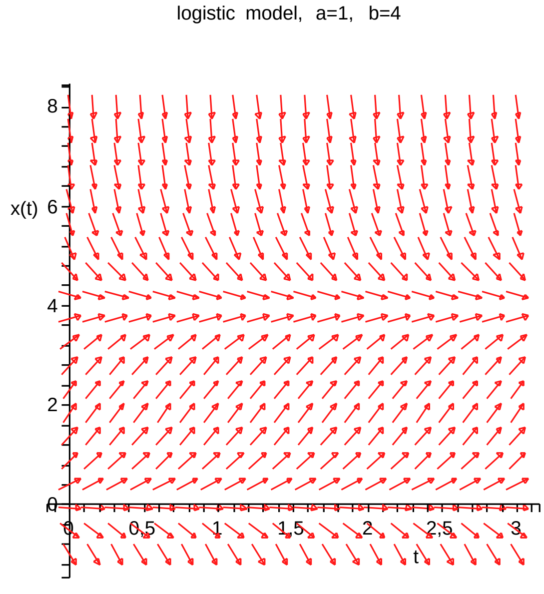

Figure 3: Logistic model

Exercice VII (Ricatti, Bernoulli and Lagrange equations)

Solve the following ordinary differential equations:

$$\dot{y} = y - \sqrt{y}\,, \qquad \dot{y} = y^2 - xy + 1\,,$$

$$y = x\dot{y} - \dot{y}^3\,, \qquad (y - x\dot{y})^2 = 1 + \dot{y}^2\,.$$

Exercice VIII (Envelopes)

Find the envelopes of the following sets of curves of $\mathbb{R}^2$:

1) $3tx - 2y = t^3$,

2) $(3t^2 - t^6)x + (3t^4 - 1)y = 2t^3$.

Exercice IX (Poincaré's Theorem)

Show the following

Theorem 1 *Let U be an open subset of $\mathbb{R}^n$, $n \geq 1$. Assume that U is star-shaped. Show that a differential form $\omega = \sum_{i=1}^{n} a_i dx_i$ of class C^1 on U is exact iff*

$$\forall i, j \in \{1, \ldots, n\}, \quad \frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}.$$

The open set U is said to be star-shaped if there exists $x_0 \in U$ such that all the segments $[x_0, x]$ joining x_0 to a point x of U are contained in U.

Exercice X (A race) Consider four horses G, T, M and P located at the ends of a square with side $L = 1$. Assume that they are moving with the same constant speed V and that G, T, M and P are attracted by T, M, P and G respectively. Each horse moves towards its favorite counterpart. The aim of the problem is to calculate the distance which is performed by any of the horses at the end of the race which will stop when the distance between any two horses is less that $d = \frac{L}{500}$.
Symetry arguments allow to consider one of the trajectories, since the others will be deduced after a rotation of angle $\dfrac{\pi}{4}$ and center O, which is also the center of the square.
A natural choice is to take O as the origin of the reference frame while the abscissa axis will be orthogonal to one side of the square. If $G(t)$ denotes the position of G at time t, let:

$$\rho(t) = OG, \quad \theta(t) = (\vec{x}, \overrightarrow{OG}).$$

1. Write the affix $z(t)$ of the point G in trigonometric form.
2. Let z_T denote the affix of T. Show that $z_T = iz$.
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4. Calculate the derivative: $\dot{z}(t) := \dfrac{dz}{dt}$. By definition, $\dot{z}(t)$ is the affix of the velocity of G in the reference frame $\mathcal{R} = (O; \vec{x}, \vec{y})$.
5. Show that there exists a real-valued function $k(t)$ such that:
$$\dot{z}(t) = k(t)(z_T(t) - z(t)). \tag{4}$$
Hint: Notice that the movement of G may be characterized in terms of the movement of T.
6. Show that:
$$\frac{\dot{\rho}}{\rho} = -\dot{\theta}. \tag{5}$$
Hint: Write down both the real part and the imaginary part in (4).
7. Integrate the differential equation (5).
8. Compute $\rho(0)$ and $\theta(0)$ in terms of the data. Conclude that the equation of the trajectory of the horse G reads:
$$\rho = \frac{\sqrt{2}}{2} e^{(\frac{\pi}{4} - \theta)}. \tag{6}$$
9. Sketch the graph of the curve (6) when $\theta \in [\frac{\pi}{4}, 2\pi]$.
10. The formula (6) yields the modulus ρ in terms of the argument θ. In this question, we aim to express ρ and θ in terms of the time t.
    (a) Compute the modulus $|\dot{z}(t)|$ and show that:
    $$(\dot{\rho})^2 + \rho^2 \dot{\theta}^2 = V^2.$$
    (b) Taking advantage of (5) and (6), show that θ solves the following differential equation:
    $$e^{-\theta} \dot{\theta} = \pm e^{-\frac{\pi}{4}} V. \tag{7}$$
    (c) Integrate (7) and show that:
    $$\theta(t) = \frac{\pi}{4} - \ln(1 \mp Vt), \quad \rho(t) = \frac{\sqrt{2}}{2}(1 \mp Vt). \tag{8}$$
    (d) Give the expression of the modulus $|z_T - z|$ in terms of t. What does this number stand for?
11. When does the race end? What is the distance run by each horse?