# ORDINARY DIFFERENTIAL EQUATIONS – F01, L3

## Exercises – Week 2-3.

# 1 Around Gronwall Lemma

## Exercise I (Osgood Theorem of uniqueness)

Let $I$ be an interval of $\mathbb{R}$, and $f\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ a continuous function. Let $\Omega$ be an open subset of $\mathbb{R}^n$ endowed with the euclidean norm, $t_0 \in I$, $x_0 \in \Omega$. We suppose that

$$\forall (t, x_1, x_2) \in I \times \Omega \times \Omega, \quad |f(t, x_1) - f(t, x_2)| \leq \omega(|x_1 - x_2|)$$

where $\omega \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$\forall \sigma > 0, \omega(\sigma) > 0, \tag{1}$$

$$\forall \alpha > 0, \int_0^\alpha \frac{1}{\omega(\sigma)} d\sigma = +\infty. \tag{2}$$

Let $x_1, x_2\colon I \to \Omega$ be two differentiable functions solutions of the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(t_0) = x_0. \end{cases}$$

Show that $x_1 = x_2$.

(*Hint:* use $r(t) := |x_1(t) - x_2(t)|^2$.)

## Exercise II (Gronwall Lemma)

**Lemma 1** *[Gronwall Lemma] Let $T > 0, c \geq 0$. If the functions $\phi, \psi \in \mathcal{C}^1([0,T); \mathbb{R}_+)$ satisfy the inequality*

$$\forall t \in [0, T), \quad \phi(t) \leq c + \int_0^t \psi(s)\phi(s) ds,$$

*then*

$$\forall t \in [0, T), \quad \phi(t) \leq c e^{\int_0^t \psi(s) ds}.$$

1) Set $h(t) := c + \int_0^t \psi(s)\phi(s)ds$ and $\theta(t) := h'(t) - \psi(t)h(t)$. Show that $\theta \in \mathcal{C}([0,T); \mathbb{R}_+)$ is non-positive.

2) Solve the ODE

$$\begin{cases} \dot{y}(t) - \psi(t)y(t) = \theta(t), \\ y(0) = c. \end{cases}$$

3) Prove the Gronwall Lemma.

### Exercise III (Comparison of quasi-solutions to ODEs)

Let $\dfrac{dx}{dt} = f(t,x)$ denote some differential equation in $\mathbb{R}^n$ where $f$ is a $k$-Lipschitz continuous function on $[0,T] \times \mathbb{R}^n$. We assume that there exist two real constants $\varepsilon_1, \varepsilon_2 \geq 0$ and two differentiable applications

$$x_i : \ I := [0,T] \to \mathbb{R}^n, \quad i = 1,2$$

such that $t \mapsto f(t, x_i(t))$ is continuous on $I$ and:

$$\left\| \frac{dx_i(t)}{dt} - f(t, x_i(t)) \right\| \leq \varepsilon_i, \quad \forall t \in I, \quad i = 1,2.$$

We set $x_i^0 = x_i(0)$, $i = 1,2$.

1) Show that:

$$\forall t \in I, \quad \|x_1(t) - x_2(t)\| \leq \|x_1^0 - x_2^0\| e^{kt} + (\varepsilon_1 + \varepsilon_2) \frac{e^{kt} - 1}{k}.$$

*Hint:* Set $\phi = \|x_1 - x_2\|$ and apply (an improved) Gronwall Lemma to $\phi$.

2) Discuss the result in the case $\varepsilon_1 = \varepsilon_2 = 0$.

## 2 The Cauchy Lipschitz theorem

### Exercise IV (Approximation of solution)

Let $q \in \mathcal{C}(\mathbb{R})$ be a bounded function. The aim of the exercise is the approximation of the solution to the ODE

$$\begin{cases} \ddot{x}(t) - q(t)x(t) = 0, \\ x(0) = 0, \dot{x}(0) = 1. \end{cases} \tag{3}$$

1) Show that (3) is equivalent to the system

$$\begin{cases} \dot{X}(t) = F(t, X(t)), \\ X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{cases} \tag{4}$$

where

$$F(t,X) = \begin{pmatrix} 0 & 1 \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \forall t \in \mathbb{R}, \quad \forall X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

*Hint:* Set $X(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$

2) Apply the Cauchy-Lipschitz Theorem to the problem (4)

3) We define the sequence $(X_n)$ by the recursion

$$\begin{cases} X_0(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ X_{n+1}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \displaystyle\int_0^t F(s, X_n(s))ds. \end{cases}$$

Show that, for $\alpha > 0$ small enough, the sequence $(X_n)$ is converging in $\mathcal{C}([-\alpha, \alpha]; \mathbb{R}^2)$ to the solution of (4).

4) We suppose that $q(t) = -1$ for all $t \in \mathbb{R}$. Give the solution to (3) and compute $X_3(t)$. What do you notice?

## Exercise V (Autonomous ODE - 1)

Let $f: \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function. Suppose that $a \in \mathbb{R}$ is a singular point for $f$, i.e. $f(a) = 0$. Let $x$ be a solution of the Cauchy Problem

$$\begin{cases} \dot{x}(t) = f(x(t)), \\ x(0) = x_0, \end{cases}$$

defined on the interval $I$.

1) Show that $x(t) - a$ has a constant sign.

2) Application: compute all the solutions of the Cauchy Problem

$$\begin{cases} \dot{x} = x(1-x), \\ x(0) = x_0 \in \mathbb{R}. \end{cases}$$

## Exercise VI (Autonomous ODE - 2)

Let $f: \mathbb{R}^N \to \mathbb{R}^N$ be a locally Lipschitz continuous function. Let $x$ be the maximal solution of the Cauchy Problem

$$\begin{cases} \dot{x}(t) = f(x(t)), \\ x(0) = x_0, \end{cases}$$

defined on the interval $I$.
Suppose that there exists $\tau_1 \neq \tau_2 \in I$ such that $x(\tau_1) = x(\tau_2)$ ($x$ is non-injective): show that $I = \mathbb{R}$ and that $x$ is periodic.

## Exercise VII (The pendulum)

The differential equation which governs the motion of a (rigid) pendulum is

$$\ddot{x} + \sin(x) = 0. \tag{5}$$

It has to be completed with the initial data

$$x(0) = x_0, \dot{x}(0) = y_0. \tag{6}$$

We want to show that, for a certain range of data $x_0$, $y_0$, the problem (5)-(6) has periodic solutions.

1) Write the problem (5)-(6) as a problem of order one in $\mathbb{R}^2$:

$$\begin{cases} \dot{X} = F(X), \\ \\ X(0) = X_0, \end{cases} \tag{7}$$

where $X(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$ and give the expression of $F$.

2) Apply the Cauchy-Lipschitz Theorem to (7).

3) Let $x$ be a solution to (5)-(6) defined on $\mathbb{R}$. Set $\lambda := y_0^2/2 - \cos(x_0)$ and let $\Gamma$ be the curve $y^2/2 - \cos(x) = \lambda$ in $\mathbb{R}^2$. Show that, for all $t \in \mathbb{R}$,

$$X(t) \in \Gamma.$$

4) We suppose $x_0 \in (-\pi, \pi)$, $y_0^2/2 - \cos(x_0) < 1$. We *admit* that, in that case, $\Gamma$ is a closed smooth curve, i.e. $\Gamma = \gamma([0, L])$ where $L > 0$, and $\gamma \in \mathcal{C}^1([0, 1]; \mathbb{R}^2)$ satisfies $|\dot{\gamma}(t)| = 1$ for all $t \in [0, T]$ and $\gamma(0) = \gamma(1)$. (See)

4-a) Show that $F(X) \neq 0$ for all $X \in \Gamma$.

4-b) Show that $\min_{X \in \Gamma} |F(X)| > 0$. What do you deduce on $|\dot{X}|$?

4-c) Show that there exists $t_1 \neq t_2 \in \mathbb{R}$ such that $X(t_1) = X(t_2)$.

4-d) By use of the Cauchy-Lipschitz Theorem, deduce that $x$ is periodic.

# 3 Existence and uniqueness for ODEs

## Exercise VIII

1) Show that the functions $t \mapsto [(t - \alpha)^+]^2/2$, $\alpha > 0$ are all solutions of the Cauchy Problem

$$\begin{cases} \dot{x}(t) = \sqrt{|x(t)|}, \\ x(0) = 0, \end{cases}$$

on $\mathbb{R}$. Discuss this result.

2) Show that, if $f \colon \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $f(x_0) \neq 0$ at the point $x_0 \in \mathbb{R}$, then the Cauchy Problem

$$\begin{cases} \dot{x}(t) = f(x(t)), \\ x(0) = x_0, \end{cases}$$

has a unique local solution.

3) Let $c_0(\mathbb{N})$ denote the Banach space of all sequences of reals $u = (u_n)$ converging to 0 at $n \to +\infty$ endowed with the norm

$$\|u\|_\infty := \sup_{n \geq 0} |u_n|.$$

3-a) Show that the mapping $f \colon u = (u_n) \mapsto (\sqrt{|u_n|} + 1/n)$ is continuous from $c_0(\mathbb{N})$ into $c_0(\mathbb{N})$.

3-b) Show, by direct integration, that, for every $t > 0$, the value $u(t)$ of the solution to the Cauchy Problem

$$\begin{cases} \dot{u}(t) = f(x(t)), \\ u(0) = 0, \end{cases}$$

is not in $c_0(\mathbb{N})$. Discuss the result.