ORDINARY DIFFERENTIAL EQUATIONS – F01, L3

Exercises – Week 2-3. II

1 Time of existence

Exercise I (Osgood Theorem of existence)

Let I be an interval of $\mathbb{R}$, and $f\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ a continuous function, locally Lipschitz continuous in the second variable. We suppose that there exists a function $a \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+^*)$ such that

$$\forall (t,x) \in I \times \mathbb{R}^n, \quad |f(t,x)| \leq a(|x|)$$

and that a satisfies

$$\int_0^{+\infty} \frac{1}{a(\sigma)} d\sigma = +\infty.$$

Show that, for $x_0 \in \mathbb{R}$, the maximal solution of the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(t_0) = x_0 \end{cases}$$

are defined on I.

(*Hint:* use $r(t) := |x(t)|^2$.)

Exercise II (Global solution)

1) Show that, in exercise IV "Approximation of solutions" of weeks 2-3, there is no need to suppose q bounded to have global solutions.

2) Generalize the conclusion to the linear ODE

$$\dot{X}(t) = A(t)X + B(t)$$

with $A\colon \mathbb{R} \to \mathcal{M}_N(\mathbb{R})$ and $B\colon \mathbb{R} \to \mathbb{R}^N$ continuous.

3) Show that, in exercise VII "The pendulum" of weeks 2-3, any solution is global.

Exercise III (Lyapunov function)

For $x, y \in \mathbb{R}^N$, we denote by $x \cdot y$ the scalar product and by $|x|$ the euclidean norm.

1) Let $v \in \mathcal{C}^1(\mathbb{R}^N; \mathbb{R}^N)$ satisfy

$$\forall x \in \mathbb{R}^N, \quad v(x) \cdot x \leq 0.$$

Show that, given $x_0 \in \mathbb{R}$, there exists a unique global solution to the Cauchy Problem

$$\begin{cases} \dot{x}(t) = v(x(t)), \\ x(0) = x_0. \end{cases}$$

Hint: Show that $t \mapsto |x(t)|^2$ is non-increasing.

2) Let $v \in \mathcal{C}^1(\mathbb{R}^N;\mathbb{R}^N)$ satisfy

$$\forall x \in \mathbb{R}^N, \quad |x| = 1 \Rightarrow v(x) \cdot x < 0. \tag{1}$$

Show that, given any starting point x_0 in the unit ball $B(0,1)$, the maximal solution of the the Cauchy Problem

$$\begin{cases} \dot{x}(t) = v(x(t)), \\ x(0) = x_0, \end{cases}$$

stays in $B(0,1)$.

Deduce that the solutions are global.

3) Show that this remains true if (1) is replaced by

$$\forall x \in \mathbb{R}^N, \quad |x| = 1 \Rightarrow v(x) \cdot x \leq 0.$$

Hint: For $\varepsilon > 0$ small, consider $v_\varepsilon(x) := v(x) - \varepsilon x$, use the theorem of continuity of the solutions with respect to parameters.