ORDINARY DIFFERENTIAL EQUATIONS – F01, L3

Exercises – Week 4-6.

1 Explicit computations

Exercise I (First-order linear equations) Solve the following differential equations

1. $\dot{y} - y = e^t$,
2. $t\dot{y} - 2y = 0$.

Exercise II (Second-order linear equations)

Solve the following equations

1. $\ddot{y} + 4\dot{y} + 3y = t$,
2. $\ddot{y} + 9y = (t+1)e^t$,
3. $\ddot{y} - 7\dot{y} + 6y = e^t$,
4. $\ddot{y} + 4y = \cos(2t)$.

Exercise III (First-order system)

1. Solve the system:
$$x_1' = x_2, \quad x_2' = x_1.$$
2. Solve $y' = Ay$ when A is defined by:
$$A = \begin{pmatrix} 4 & -4 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

2 Trajectories, flow for linear equations

Exercise IV (Approximation of solution)

Let d be a positive integer. We admit that, for every $A \in \mathcal{M}_d(\mathbb{R})$,

$$\lim_{n\to+\infty} \left(I_d + \frac{A}{n}\right)^n = e^A.$$

1. Let $A \in \mathcal{M}_d(\mathbb{R})$ and $x_0 \in \mathbb{R}^d$, $T > 0$ be given. Let x be the solution of the differential equation
$$\begin{cases} \dot{x}(t) &= Ax(t) \\ x(0) &= x_0 \end{cases} \tag{1}$$
What is the value of $x(T)$? (use the exponential function)

2. To compute the value $x(T)$, we use the method of Euler: divide the interval $[0,T]$ in n intervals of size $h := T/n$. Let $k \in \{1, \cdots, n\}$. We approximate the value $x(kh)$ of the solution x of (1) at time kh by the real x^k where x^k is defined by the recursion

$$x^0 = x_0, \qquad x^{k+1} = x^k + hAx^k.$$

   Compute x^n and show that it converges to $x(T)$ when $n \to +\infty$.

3. a) Suppose $d = 2$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

   Draw the image by A of any vector of $\mathbb{R}^2$.

   b) Choose any point $x_0 \in \mathbb{R}^2$ and draw (approximatively) the successive points x^k defined in 2).

   c) Compute e^{tA} by using the formula

$$e^{tA} = \sum_{k=0}^{\infty} t^k A^k / k!.$$

   d) Choose the same point x_0 as in I-3-b) and draw the curve $\{e^{tA}x_0, \ 0 < t < T\}$. Compare with 3-b).

4. Repeat the questions of 3) in the case where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

5. Repeat the questions of 3) in the case where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

3 Equations $\ddot{y} + qy = 0$

In the whole section, q is a continuous function $\mathbb{R} \to \mathbb{R}$. Each exercise is independent on the other.

Exercise V ($q - 1 \in L^1(\mathbb{R})$)

We suppose that $q = 1 - p$ where

$$\int_{\mathbb{R}} |p(t)| dt < +\infty.$$

Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ be a solution to $\ddot{y} + qy = 0$. Show that φ is bounded on $\mathbb{R}$.

Hint: φ is solution to $\ddot{y} + y = f(t)$ with $f(t) = p(t)\varphi(t)$. Use the resolvant formula and Gronwall Lemma.

Exercise VI ($q \in L^1(\mathbb{R})$)

We suppose that

$$\int_{\mathbb{R}} |q(t)| dt < +\infty.$$

1) Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ be a bounded solution to $\ddot{y} + qy = 0$. Show that

$$\lim_{t\to+\infty} \dot{\varphi}(t) = 0.$$

2) Show that there exists some unbounded solutions to $\ddot{y} + qy = 0$.

Hint: proceed by contradiction and study the wronskian of two solutions.

Exercise VII (Zeros of the solutions)

1) Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ be a solution to $\ddot{y} + qy = 0$. Show that the zeros of φ are isolated (*i.e.* if $t_0 \in \mathbb{R}$ is such that $\varphi(t_0) = 0$, there exists $\alpha > 0$ such that $\varphi(t) \neq 0$ for every $t \in (t_0 - \alpha, t_0 + \alpha) \setminus \{t_0\}$).

2) Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ be a solution to $\ddot{y} + qy = 0$. Let $t_1 < t_2 \in \mathbb{R}$ be such that

$$\varphi(t_1) = \varphi(t_2) = 0, \quad \varphi > 0 \text{ on } (t_1, t_2).$$

2-a) Let $\bar{q}\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $q \leq \bar{q}$. Let ψ be a solution to $\ddot{y} + \bar{q}y = 0$. Show that ψ has a zero in $[t_1, t_2]$.

Hint: study the wronskian $w := \varphi\dot{\psi} - \dot{\varphi}\psi$.

2-b) (Application) We suppose $q(t) \geq m$ for all $t \in \mathbb{R}$, with $m > 0$. Show that

$$t_2 - t_1 \leq \frac{\pi}{\sqrt{m}}.$$

4 Linear ODEs and linear algebra

Exercise VIII

Let $A \in \mathcal{M}_n(\mathbb{C})$. The aim of the exercise is to prove that the following conditions are equivalent:

(a) For all $b\colon \mathbb{R}_+ \to \mathbb{C}^n$ continuous and bounded, there exists a unique *bounded* application $x \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{C}^n)$ such that $x'(t) = Ax(t) + b(t)$ for every $t \geq 0$.

(b) $\mathrm{Sp}(A) \subset \{\lambda \in \mathbb{C}; \Re\lambda > 0\}$.

1) Suppose *(a)*. Let $\lambda \in \mathrm{Sp}(A)$ and $x_0 \in \mathbb{C}$ such that $Ax_0 = \lambda x_0$. What is the solution to $\dot{x} = Ax$ with initial condition x_0? Deduce that *(a)⇒(b)*.

2) Suppose *(b)*. Let $b\colon \mathbb{R}_+ \to \mathbb{C}^n$ be a continuous and bounded function. We admit that, if *(b)* is satisfied, then there exists $C, \varepsilon > 0$ such that

$$\forall s \in \mathbb{R}_+, \quad |e^{-sA}| \leq Ce^{-s\varepsilon}$$

where $|\cdot|$ is any given norm on $\mathcal{M}_n(\mathbb{C})$.

2)-a) Show the uniqueness part of question *(a)*.

2)-b) Write the variation of constant formula for the solution of the Cauchy Problem $x'(t) = Ax(t) + b(t)$, $x(0) = x_0$.

2)-c) Show that $\displaystyle\int_0^\infty e^{-sA}b(s)ds$ is finite.

2-d) Give the value of x_0 for which the Cauchy Problem $x'(t) = Ax(t) + b(t)$, $x(0) = x_0$ has a solution bounded on $\mathbb{R}_+$.

Exercise IX

Consider the linear system:

$$x'(t) = A(t)x(t)$$

where $A \in C(\mathbb{R}; \mathcal{M}_n(\mathbb{R}))$. Let φ be some nonzero solution of this sytem and set

$$\gamma = \limsup_{t\to+\infty} \frac{1}{t} \ln \|\varphi(t)\|, \quad -\infty \leq \gamma \leq +\infty.$$

1. Show that γ does not depend on the choice of the norm $\|\cdot\|$ of $\mathbb{R}^n$.
2. Assume that the matrix $A(t)$ has bounded coefficients. Show that γ is finite. *Hint:* use Gronwall identity.
3. Assume that A is constant and diagonalizable. Show that γ is the real part of some eigenvalue of A.

Exercise X

Let $E \subset \mathbb{R}^n$ be some p-dimensional vectorial space, $1 \leq p \leq n$, and let A be some $n \times n$ real matrix that preserves E. Show that if $x(t)$ is a solution of the differential equation $x' = Ax$ such that $x_0 = x(0) \in E$, then

$$\forall t \in \mathbb{R} : \, x(t) \in E.$$