ORDINARY DIFFERENTIAL EQUATIONS – F01, L3

Exercises – Week 6.

The following exercises are devoted to the resolution of the ode

$$y'' + p(x)y' + q(x) = 0 \tag{1}$$

by means of expansion in power series.

We will suppose that Eq. (1) has a *regular singularity* at $x = 0$, i.e.

$$xp(x) = \sum_{n\geq 0} p_n x^n, \quad x^2 q(x) = \sum_{n\geq 0} q_n x^n$$

with non-trivial radius of convergence.

We also recall that the *indicial equation* associated to (1) is the equation

$$r(r-1) + p_0 r + q_0 = 0. \tag{2}$$

We denote by r_1 and r_2 the roots of (2), with the convention $\Re(r_1) \geq \Re(r_2)$.

Exercise I ($r_1 - r_2 \notin \mathbb{N}$)

Compute the expansion in power series of two independent solutions of the following equations:

1. $3x(1+x)y'' + y' - 6y = 0$,
2. $x^2y'' + xy' + (x-2)y = 0$,
3. $2x^2y'' + x(1+2x)y' - y = 0$,
4. $4x(1-x)y'' + 2(1-2x)y' + y = 0$.

Answers:

1. $y_1(x) = 1 + 6x + \frac{9}{2}x^2$, $y_2(x) = x^{2/3}\left(1 + \frac{4}{3}x + \frac{2}{9}x^2 + \cdots\right)$,
2. Bessel,
3. $y_1(x) = x\left(1 - \frac{2}{5}x + \frac{4}{35}x^2 + \cdots\right)$, $y_2(x) = x^{-1/2}\left(1 - x + \frac{1}{2}x^2 + \cdots\right)$,
4. $y_1(x) = \sqrt{x}$, $y_2(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$.

Exercise II ($r_1 - r_2 \in \mathbb{N}$)

Compute the expansion in power series of two independent solutions of the following equations:

1. $xy'' + xy' - y = 0$,
2. $xy'' + (1-x)y' + 2y = 0$,
3. $x^2y'' + x^2y' - 2y = 0$,

4. $xy'' + y' + y = 0$.

Answers:

1. $y_1(x) = x$, $y_2(x) = x\ln(x) + \left(1 + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}x^n}{n!(n-1)}\right)$,
2. $y_1(x) = 1 - 2x + \frac{1}{2}x^2$, $y_2(x) = y_1(x)\ln(x) + 5x - \frac{9}{4}x^2 + \frac{1}{18}x^3 + \cdots$,
3. $y_1(x) = x^{-1} - \frac{1}{2}$, $y_2(x) = x^2 - \frac{1}{2}x^3 + \cdots$
4. $y_1(x) = \sum_{n\geq 0}(-1)^n \frac{x^n}{(n!)^2}$, $y_2(x) = y_1(x)\ln(x) + \left(2x - \frac{3}{4}x^2 + \frac{11}{108}x^3 + \cdots\right)$.

Exercise III (Independence of the solutions)

We denote by $W(u,v) = uv' - u'v$ the wronskian of two functions u and v.

1. Case $r_1 - r_2 \notin \mathbb{N}$:

   Let $y_1(x) = x^{r_1}\sum_{n\geq 0} a_n x^n$ and $y_2(x) = x^{r_2}\sum_{n\geq 0} b_n x^n$ be two solutions of (1). Show that $W(y_1, y_2)$ has the form
   $$x^{r_1+r_2-1}\sum_{n\geq 0} c_n x^n$$
   where $c_0 = (r_2 - r_1)a_0 b_0$. Deduce that y_1 et y_2 are two independent solutions of Eq. (1).

2. Case $r_1 - r_2 \in \mathbb{N}$:

   Suppose now that $y_1(x) = x^{r_1}\sum_{n\geq 0} a_n x^n$ and $y_2(x) = a\ln(x)y_1(x) + x^{r_2}\sum_{n\geq 0} b_n x^n$ are two solutions of Eq. (1).
   Show that $W(y_1, ay_1 \ln) = ay_1(x)^2/x$ then show that $W(y_1, y_2)$ has an expansion of the form
   $$W(y_1, y_2) = ax^{2r_1-1}\sum_{n\geq 0} c_n x^n + x^{r_1+r_2-1}\sum_{n\geq 0} d_n x^n$$
   where $c_0 = a_0^2$ and $d_0 = (r_1 - r_2)a_0 b_0$.

   Show that y_1 and y_2 are independent.

Exercise IV (Frobenius Method)

We develop here the method used by Frobenius to derive the solutions of (1) in the case $r_1 - r_2 \in \mathbb{N}$ (it dates back to year 1873).

1) Show by formal comptations that $y(x) = x^r \sum_{n\geq 0} c_n x^n$ is solution to (1) if: $P(r)c_0 = 0$ and, for all $n \geq 1$,
$$P(r+n)c_n + (\text{function of})(p_k, q_k, c_k; k < n) = 0 \tag{3}$$
where $P(r) = r(r-1) + p_0 r + q_0$.

2) We suppose that r is such that $P(r+n) \neq 0$ for all $n \geq 1$ and solve (3) with datum $c_0 = 1$. Let $y_*(x,r)$ denote the function we thereby obtain. Check that

$$y_*'' + p(x)y_*' + q(x)y_* = x^{r-2}P(r). \tag{4}$$

3) We suppose $r_1 = r_2$.

3-a) Show that $y_*(x, r_1)$ and $\dfrac{\partial y_*}{\partial r}(x, r_1)$ are two solutions of (1)

3-b) Show that $\dfrac{\partial y_*}{\partial r}(x, r_1) = y_1(x)\ln(x) + x^{r_2}\sum_{n\geq 0} b_n x^n$.

3-c) Show that y_1 and y_2 are independent solutions of (1) (use Exercise III).

4) We suppose $r_1 = r_2 + m$ with $m \in \mathbb{N}^*$. We set $z_*(x,r) = (r - r_2)y_*(x,r)$.

4-a) Show that $y_*(x, r_1)$ and $\dfrac{\partial z_*}{\partial r}(x, r_2)$ are two solutions of (1).

4-b) (Tricky question) Show that $\dfrac{\partial z_*}{\partial r}(x, r_2) = ay_1(x)\ln(x) + x^{r_2}\sum_{n\geq 0} b_n x^n$ where $a \in \mathbb{R}$.

4-c) Show that y_1 and y_2 are independent solutions of (1) (use Exercise III).