ORDINARY DIFFERENTIAL EQUATIONS – F01, L3

Exercises – Week 7-8.

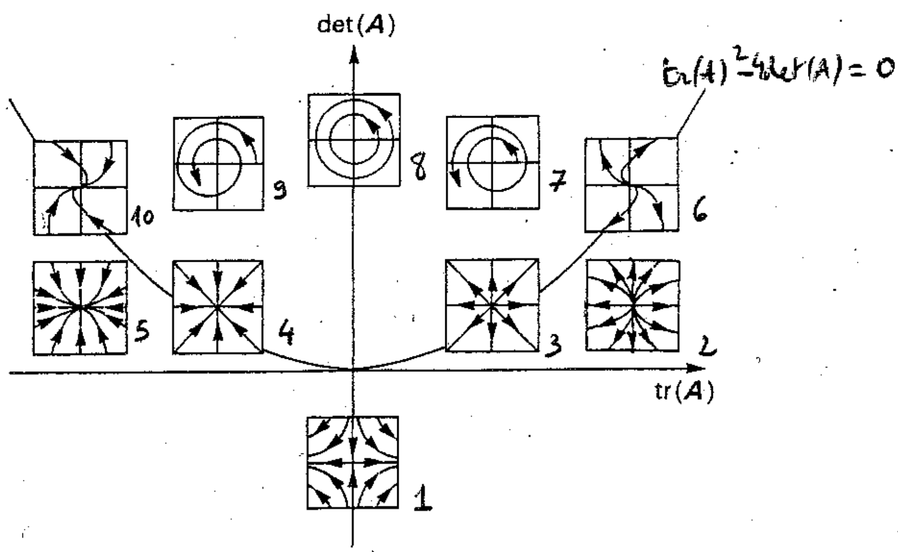
Exercise I (Phase portraits in the plane)

On the following graph are pictured different phase portraits for the system $\dot{x} = Ax$, $A \in \mathcal{M}_2(\mathbb{R})$.

1) For each of these pictures:

- give the nature of the stationary point $(0,0)$ (foci, node, stable, unstable, etc.),
- discuss the eigenvalues, wether the matrix A is diagonazible on $\mathbb{R}$ or not,
- give an example of matrix A such that the trajectories of $\dot{x} = Ax$ have the behaviour described on the picture.

2) Two phase portraits are missing on the following graph. Which ones ?



Exercise II (The pendulum)

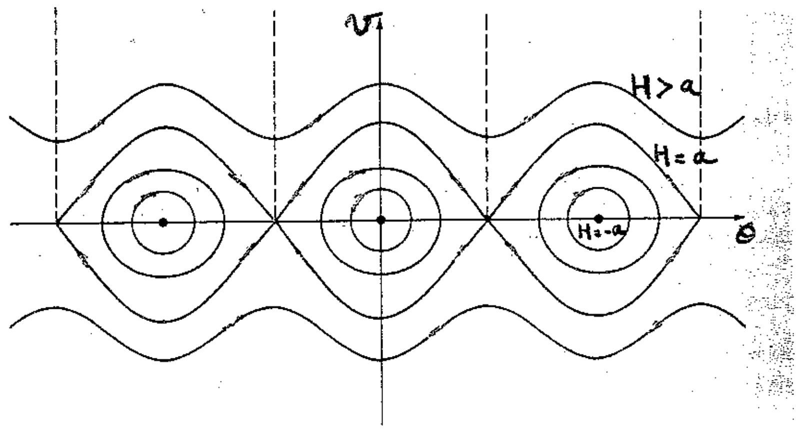
Consider a pendulum of lenght l.

1) Let g denote the acceleration of the gravity and set $a := g/l > 0$. Show that, if there is no friction, the equation of the motion of the pendulum is

$$\theta'' = -a\sin(\theta). \tag{1}$$

2) We set $v = \theta'$. Show that (1) can be rewritten as an hamiltonian system. Give the hamiltonian H.

3) On the following drawing are pictured the level-set curves $H(\theta, v) = \text{Cst}$? What are the trajectories of (1) in the phase plane (θ, v)?



4) Interpret on your drawing the following configurations:

1. The pendulum is pushed from the position $\theta(0) = 0$ with a small initial angular speed $v(0)$ ($v(0) < \sqrt{2a}$).
2. The pendulum is pushed from the position $\theta(0) = 0$ with a large initial angular speed $v(0)$ ($v(0) > \sqrt{2a}$).
3. The pendulum is pushed from the position $\theta(0) = \pi$ with an initial angular speed $v(0) > 0$.

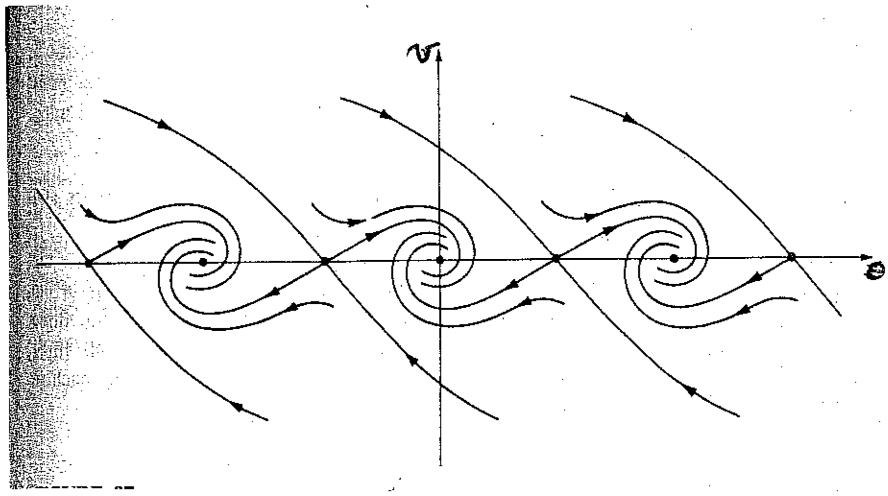
5) Show that the stationary points of the equation (the zeros of F) are the points $(k\pi, 0)$, $k \in \mathbb{Z}$.. From a look at the phase portrait, which ones seem stable, which ones unstable?

6) We now consider the case where the friction is taken into account: the equation of the pendulum becomes

$$\theta'' = -\alpha\theta' - a\sin(\theta)$$

with $\alpha > 0$. Write the equation as a first order system in the variables (θ, v) and show that the stationary points still are the points $(k\pi, 0)$, $k \in \mathbb{Z}$. Justify the fact that, for α large enough, the points $(2k\pi, 0)$ are stable (use Lagrange stability Theorem).

7) A drawing of the phase portrait (for α large) is the following one. Discuss it.



Exercise III (Lyapunov functional)

Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function on $\mathbb{R}^n$. We suppose that the ordinary differential equation

$$x' = f(x) \tag{2}$$

admits a Lyapunov functional, that is to say: there exists a continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\forall x \in \mathbb{R}^n, \quad dV(x) \cdot f(x) \leq 0.$$

1) Let (x, I) be a solution of (2). Show that $t \mapsto V(x(t))$ is non-increasing on I.

2) Let x_0 be an equilibrium of (2): $f(x_0) = 0$. Assume furthermore that x_0 is a strict local minimum of V. Then there exists $r_1 > 0$ such that, for all $0 < r < r_1$, $\alpha_r := \min_{|x-x_0|=r} V(x) > V(x_0)$.

2-a) Let $0 < r < r_1$. Show that $U_r := \{x \in \mathbb{R}^n, V(x) < \alpha_r\}$ is a neighborhood of x_0 and that all the trajectories starting from U_r stay in $B(x_0, r)$.

2-b) Show that the equilibrium x_0 is stable.

3) With the same hypotheses and notations than in question 2), we additionally suppose that

$$\forall x \neq x_0, \quad dV(x) \cdot f(x) < 0.$$

We want to prove that x_0 is asymptotically stable.

3)-a) Let $r \in (0, r_1)$ be fixed. Justify the fact that the flow $\Phi(x, t)$ of (2) is defined on $U_r \times \mathbb{R}_+$.

3-b) For $x \in U_r$, we denote by $\omega(x)$ the ω-limit set of x; this is the set of the adherence values of $t \mapsto \Phi(x, t)$ when $t \to +\infty$, *i.e.*

$$\omega(x) = \{y \in \mathbb{R}^n, \exists (t_n) \in \mathbb{R}_+, t_n \to +\infty \text{ and } \lim_{n \to +\infty} \Phi(x, t_n) = y\}.$$

Show the La Salle Principle: if $y \in \omega(x)$ then $t \mapsto V(\Phi(y, t))$ is constant.

3-c) Show that $\omega(x) \subset \{y \in \mathbb{R}^n, dV(y) \cdot f(y) = 0\}$.

3-d) Show that $\omega(x) = \{x_0\}$ and conclude.

Exercise IV (Equilibrium in a conservative field)

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathcal{C}^2$. We consider the ordinary differential equation

$$\ddot{x}(t) = -\nabla\phi(x). \tag{3}$$

1) Write (3) as a Hamiltonian system (*Hint:* what is the total energy?).

2) Assume that $x_0 \in \mathbb{R}^n$ is a local strict minimum of ϕ. By use of exercise III, show that x_0 is a stable equilibrium of (3) (*Hint:* find a Lyapunov function).