Ex. 9.

6- a) Soit $z = (x;y) \in D(z_0; \frac{1}{2}|z_0|)$. On a $|z - z_0| < \frac{1}{2}|z_0| = \frac{1}{2}|x_0|$. Comme $|Re(z-z_0)| \leq |z - z_0|$, on a $|x - x_0| = |Re(z-z_0)| < \frac{1}{2}|x_0|$ et $x_0 - \frac{1}{2}|x_0| < x < x_0 + \frac{1}{2}|x_0|$.
Comme $x_0 < 0$, $x \leq x_0 + \frac{1}{2}|x_0| = x_0 - \frac{1}{2}x_0 = \frac{x_0}{2} < 0$.
Donc $x < 0$ et $|x| = -x \geq -\frac{x_0}{2} = \frac{|x_0|}{2}$. On a donc

$$\left|\frac{y}{x}\right| \leq |y| \frac{2}{|x_0|}.$$

b) Par a), on sait que $x < 0$. Comme $y \neq 0$, on a

$$\frac{y}{x + \sqrt{x^2+y^2}} = \frac{y}{x + |x|\sqrt{1+\frac{y^2}{x^2}}} = \frac{y}{-|x| + |x|\sqrt{1+\frac{y^2}{x^2}}} = \frac{y/|x|}{\sqrt{1+\frac{y^2}{x^2}} - 1} = \frac{y}{|x|} \frac{\sqrt{1+\frac{y^2}{x^2}}+1}{1+\frac{y^2}{x^2}-1}$$

$$= \frac{y}{|x|} \frac{|x|^2}{y^2}\left(1 + \sqrt{1+\frac{y^2}{x^2}}\right) = \frac{|x|}{y}\left(1 + \sqrt{1+\frac{y^2}{x^2}}\right).$$

c). Pour $z \in J_+ \cap D(z_0; \frac{|z_0|}{2})$ ou $z \in J_- \cap D(z_0; \frac{|z_0|}{2})$, on a, par b) et (4),

$$Arg(z) = 2 \operatorname{Arctan}\left(\frac{|x|}{y}\left(1 + \sqrt{1+\frac{y^2}{x^2}}\right)\right)$$

$$\frac{|x|}{y} \xrightarrow[(x;y)\in J_+]{(x;y)\to(x_0;0)} +\infty \qquad \frac{|x|}{y} \xrightarrow[(x;y)\in J_-]{(x;y)\to(x_0;0)} -\infty$$

$$1 + \sqrt{1+\frac{y^2}{x^2}} \xrightarrow[(x;y)\in J_+]{(x;y)\to(x_0;0)} 2 \qquad 1 + \sqrt{1+\frac{y^2}{x^2}} \xrightarrow[(x;y)\in J_-]{(x;y)\to(x_0;0)} 2$$

composition

$$Arg(z) \xrightarrow[(x;y)\in J_+]{(x;y)\to(x_0;0)} 2\frac{\pi}{2} = \pi \qquad Arg(z) \xrightarrow[(x;y)\in J_-]{(x;y)\to(x_0;0)} 2\left(-\frac{\pi}{2}\right) = -\pi$$

Car $\lim_{t\to+\infty} \operatorname{Arctan} t = \frac{\pi}{2}$ et $\lim_{t\to-\infty} \operatorname{Arctan} t = -\frac{\pi}{2}$.