Ex. 29.

2 – Calcul de $I := \int_{\gamma_2} \frac{dz}{z}$.

$$I = \underbrace{\int_0^{\pi/2} \frac{i e^{it}}{e^{it}}\, dt}_{I_1} + \underbrace{\int_{\pi/2}^{\pi} \frac{\frac{2}{\pi}(1-i)}{\frac{2}{\pi}(1-i)t + 2i - 1}\, dt}_{I_2}$$

$I_1 = \int_0^{\pi/2} i\, dt = i\frac{\pi}{2}$.

Par changement de variables $\frac{2}{\pi}t = 1 + s$, on a

$$I_2 = \int_0^1 \frac{(1-i)}{(1-i)(1+s) + 2i - 1}\, ds = (1-i)\int_0^1 \frac{ds}{s + i(1-s)}$$

$$= (1-i)\int_0^1 \frac{s + i(s-1)}{|s + i(1-s)|^2}\, ds$$

Comme $|s + i(1-s)|^2 = s^2 + (1-s)^2 = 2s^2 - 2s + 1$
$= 2\left(s^2 - s + \frac{1}{2}\right) = 2\left(\left(s - \frac{1}{2}\right)^2 + \frac{1}{4}\right)$,

$$I_2 = \frac{(1-i)}{2}\left(\int_0^1 \frac{s}{\left(s-\frac{1}{2}\right)^2 + \frac{1}{4}}\, ds\, (1+i) - i\int_0^1 \frac{ds}{\left(s - \frac{1}{2}\right)^2 + \frac{1}{4}}\right)$$

$$= \int_0^1 \frac{s}{\left(s-\frac{1}{2}\right)^2 + \frac{1}{4}}\, ds - \frac{i+1}{2}\int_0^1 \frac{ds}{\left(s-\frac{1}{2}\right)^2 + \frac{1}{4}} = \int_0^1 \frac{s - \frac{1}{2}}{\left(s-\frac{1}{2}\right)^2 + \frac{1}{4}}\, ds - \frac{i}{2}\int_0^1 \frac{2 \times 2\, ds}{(2s-1)^2 + 1}$$

$$= \underbrace{\left[\frac{1}{2}\ln\left(\left(s - \frac{1}{2}\right)^2 + \frac{1}{4}\right)\right]_0^1}_{=0} - i\left[\operatorname{Arctan}(2s - 1)\right]_0^1$$

$$= -i\, 2 \operatorname{Arctan}(1) = -i\, 2\frac{\pi}{4} = -i\frac{\pi}{2}.$$

Donc $I = I_1 + I_2 = 0$.