$$\det\left(M_{\mathcal{B};\tilde{\mathcal{B}}}(g)\right) = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}.$$

$$C_1 \leftarrow C_1 - C_2 - C_3$$

$$\det\left(M_{\mathcal{B};\tilde{\mathcal{B}}}(g)\right) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0.$$

Donc, par le cours, g n'est pas inversible.

6.5. On a
$$f(1) = 1 + 0 = 1$$
$$f(X) = X + 1$$
$$f(X^2) = X^2 + 2X.$$

Donc

$$M_{\mathcal{B};\mathcal{B}}(f) = \begin{array}{cccc} f(1) & f(X) & f(X^2) & \\ & & & 1 \\ & & & X \\ & & & X^2 \end{array} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Or a $\det\left(M_{\mathcal{B};\mathcal{B}}(f)\right) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$

Donc $rg(f)$ est 3 et f est inversible.

On a $(X-1)^2 = 1 - 2X + X^2$ donc

$$P_{\mathcal{B};\tilde{\mathcal{B}}} = \begin{array}{cccc} 1 & 2X+1 & (X-1)^2 & \\ & & & 1 \\ & & & X \\ & & & X^2 \end{array} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Or a $\det P_{\mathcal{B};\tilde{\mathcal{B}}} = 2$ et $\mathrm{com}\left(P_{\mathcal{B};\tilde{\mathcal{B}}}\right) = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & 2 & 2 \end{pmatrix}$

Donc $\left(P_{\mathcal{B};\tilde{\mathcal{B}}}\right)^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$

Par le cours

$$M_{\tilde{\mathcal{B}};\tilde{\mathcal{B}}}(f) = \left(P_{\mathcal{B};\tilde{\mathcal{B}}}\right)^{-1} M_{\mathcal{B};\mathcal{B}}(f) P_{\mathcal{B};\tilde{\mathcal{B}}} = \frac{1}{2}\begin{pmatrix} 2 & -1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{M}_{\tilde{\mathcal{B}},\tilde{\mathcal{B}}}(f) = \frac{1}{2}\begin{pmatrix} 2 & 4 & -6 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Alternative : on a

$$f(1) = 1, \quad f(2x+1) = 2x+1+2 = (2x+1) + 2 \times 1$$

$$f((x-1)^2) = (x-1)^2 + 2(x-1) = (x-1)^2 + (2x+1) - 3 \times 1.$$

Donc

$$\mathcal{M}_{\tilde{\mathcal{B}},\tilde{\mathcal{B}}}(f) = \begin{array}{c} \begin{array}{ccc} f(1) & f(2x+1) & f((x-1)^2) \end{array} \\ \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} 1 \\ 2x+1 \\ (x-1)^2 \end{array}.$$